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# OPENING-MODE CRACK IN A UNIDIRECTIONAL COMPOSITE MATERIAL

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*The plane static elastic problem of stress concentration in a unidirectional discrete infinite composite weakened by fiber breaks on a line normal to the reinforcement direction (an analog of the Griffith problem of elasticity theory) is considered. The composite is subjected to uniform stresses at infinity, and the crack edges are loaded symmetrically by the normal pressure. The problem reduces to constructing a polynomial with known values at the points of fiber breaks. The stress distribution along the line of breaks is obtained in the form of a fractional rational function of fiber number.*

**Key words:** *composite, fiber break, crack, stress concentration.*

**1.** Let the reinforcement fibers be loaded by uniform stress $\sigma$ at infinity and the crack edges be loaded symmetrically by pressure $P_j$ ($j$ is the fiber number). We assume that the fibers only work in tension and compression like one-dimensional rods and the binder only work in shear at sites parallel to the reinforcement direction. We denote the displacement of the $j$th fiber in the reinforcement direction by $v_j$ and write the equilibrium equations in dimensionless form:

$$\frac{d^2 v_j}{d y^2} + \beta^2 (v_{j-1} - 2v_j + v_{j+1}) = 0, \qquad -\infty < j < \infty. \tag{1.1}$$

Here $\beta^2 = \mu H/(Eh)$ is the ratio of the shear stiffness of the binder to the tension stiffness of the fibers, $\mu$ is the shear modulus of the binder, $E$ is Young's modulus of the fiber, and $h$ is the fiber width; the quantities having the dimension of length (the displacements $v_j$ and the $y$ coordinate along the fibers) are normalized by the binder width $H$.

The crack is a set of fiber breaks $M$ for $y = 0$ and $L + 1 \leqslant j \leqslant L + M$. The displacements vanish at the crack continuation and the following stresses are specified at the crack edges and at infinity:

$$v_j(0) = 0, \qquad -\infty < j \leqslant L, \quad L + M + 1 \leqslant j < \infty,$$

$$E\,\frac{dv_j}{dy}(0) = P_j, \quad L + 1 \leqslant j \leqslant L + M, \qquad E\,\frac{dv_j}{dy}(\infty) = \sigma.$$

This problem appears to be first studied in [1] for a free crack ($P_j = 0$). Initially, the problem was solved for a crack formed by one fiber break. The fundamental solution obtained for this case was then used to reduce the problem of $M$ fiber breaks to the following system of equations:

$$\sum_{s=L+1}^{L+M} \frac{b_s}{(j-s)^2 - 1/4} = -1, \qquad L + 1 \leqslant j \leqslant L + M. \tag{1.2}$$

Here $L + 1 \leqslant s \leqslant L + M$ are the broken fiber numbers. (Here and below, the notation of [1] is slightly changed to make all formulas uniform.) Unity on the right side of (1.2) is the dimensionless stress at infinity. The displacements of the upper edge of the crack (i.e., the half crack-opening displacement) are expressed in terms of $b_s$ as follows:

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$v_s(0) = \sigma\pi\beta b_s/\mu$. The stresses are normalized by $\sigma$. Given $b_s$, one can calculate the stress concentration in the $j$th intact fiber on the line of breaks:

$$Q_j = 1 + \sum_{s=L+1}^{L+M} \frac{b_s}{(j-s)^2 - 1/4}. \qquad (1.3)$$

Here the stresses are normalized by the stress applied at infinity.

The cases with small numbers of broken fibers ($-n \leqslant s \leqslant n$, $n = 0, 1, 2$) given in [1] make it possible, without solving system (1.2) in general form, to obtain the stress concentration in the first intact fiber:

$$Q_{n+1} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(2n+3)}{\Gamma(2n+5/2)} \qquad (1.4)$$

($\Gamma$ is the gamma function). For any number of broken fibers, the solution of system (1.2) is written in the form [2]:

$$b_j = \frac{1}{\pi} \frac{\Gamma(n-j+3/2)\Gamma(n+j+3/2)}{\Gamma(n-j+1)\Gamma(n+j+1)}. \qquad (1.5)$$

In [2], formula (1.4) is proved and the following expressions for The stress concentration in the next two fibers are obtained (as in [1], for uniform extension):

$$Q_{n+2} = (n+2)\frac{\sqrt{\pi}}{2} \frac{\Gamma(2n+3)}{\Gamma(2n+7/2)}, \qquad Q_{n+3} = (n+3)\frac{3\sqrt{\pi}}{2^3} \frac{\Gamma(2n+4)}{\Gamma(2n+9/2)}.$$

The formula for the stress in any fiber can be written using three values $Q_{n+1}$, $Q_{n+2}$, and $Q_{n+3}$:

$$Q_{n+k} = (n+k)\frac{\Gamma(k-1/2)}{\Gamma(k)} \frac{\Gamma(2n+1+k)}{\Gamma(2n+1+k+1/2)}, \qquad k \geqslant 1. \qquad (1.6)$$

Formula (1.6) is not proved in [2], but any numerical verification using (1.2) and (1.3) confirms its validity.

The solution of system (1.2) and the derivation of formulas for $Q_{n+1}$, $Q_{n+2}$, and $Q_{n+3}$ in [2] were based on the fact that the successive factors in the denominators of each row of matrix (1.2) form an arithmetical progression and the right sides are constant. It was not possible to solve the problem for variable $P_j$. Below, problem (1.2) and (1.3) is solved using the fact that the fractional rational function of $j$ on the right side of (1.3) can be decomposed into a few partial fractions. That is, if the number of partial fractions is less than twice the number of broken fibers, the problem reduces to a system of lower order than system (1.2). In the case where only stresses (1.3) are of interest, one does not need to solve system (1.2) to determine the crack-opening displacements $b_s$ (see, e.g., [11]). Below, the cases of remote tension and loading of broken fibers at the site of failure are considered separately.

**2.** We assume that the crack edges are stress-free and stress is applied at infinity. The right side of (1.3) is treated as a function of the index $j$. Reducing all the fractions to a common denominator, we find that the resulting function is a ratio of polynomials and that the roots of the denominator are equal to $L + 1/2, \ldots, L + M + 1/2$. At infinity, the stress concentration $Q_j$ tends to unity and, hence, the highest terms of the polynomials in the nominator and denominator are equal. Some roots of the nominator are known since $Q_j$ vanishes at the points $L + 1 \leqslant j \leqslant L + M$ (stress-free crack). Therefore, the stress concentration in the $j$th fiber can be written with accuracy up to the factor $j - a$:

$$Q_j = \frac{j-a}{j-L-1/2} \prod_{s=L+1}^{L+M} \frac{j-s}{j-s-1/2} \qquad (2.1)$$

($a$ is an unknown constant). System (1.2) can be simplified using (2.1). Decomposing the sum over the broken fibers on the right side of (1.3) into partial fractions, we obtain

$$Q_j - 1 = \frac{-b_{L+1}}{j-L-1/2} + \frac{b_{L+1} - b_{L+2}}{j-L-3/2} + \ldots + \frac{b_{L+M-1} - b_{L+M}}{j-L-3/2} + \frac{b_{L+M}}{j-L-M-1/2}. \qquad (2.2)$$

Decomposition of (2.1) into partial fractions for the same quantity $Q_j - 1$ yields

$$Q_j - 1 = \sum_{s=L}^{L+M} \frac{A_s}{j-s-1/2}. \qquad (2.3)$$

Since the roots of the denominator in (2.1) are simple, the coefficients of expansion (2.3) are calculated by the formulas

$$A_s = \lim_{j\to s+1/2} \frac{(j-s-1/2)(j-a)}{j-L-1/2} \prod_{r=L+1}^{L+M} \frac{j-r}{j-r-1/2}$$

$$= (s+1/2-a) \prod_{r=L+1}^{L+M} (s+1/2-r) \Big/ \prod_{r=L,r\neq s}^{L+M} (s-r). \qquad (2.4)$$

A comparison of the coefficients in (2.2) and (2.3) leads to the following two-diagonal system for the quantities $b_j$, which determine the crack-opening displacement:

$$\begin{aligned} -b_{L+1} &= A_L, \\ b_{L+1} - b_{L+2} &= A_{L+1}, \\ b_{L+2} - b_{L+3} &= A_{L+2}, \\ &\ldots\ldots\ldots\ldots \\ b_{L+M-1} - b_{L+M} &= A_{L+M-1}, \\ b_{L+M} &= A_{L+M}. \end{aligned} \qquad (2.5)$$

The right side of (2.5) contains the unknown parameter $a$. Summing all equations (2.5) and eliminating the crack-opening displacement, we obtain a linear equations for $a$. Determining the constant $a$, we insert it into formula (2.1) and obtain the stress distribution along the line of breaks. The right side of (2.5) becomes known, and the quantities $b_s$ are calculated successively. The calculations yield

$$a = L + \frac{M+1}{2}, \qquad Q_j = \frac{j-L-1/2-M/2}{j-L-1/2} \prod_{s=L+1}^{L+M} \frac{j-s}{j-s-1/2}. \qquad (2.6)$$

The expression for $a$ in (2.6) shows that the stress distribution is symmetric about the crack center. Let $L=-n-1$, $M=2n+1$, and $j=n+k$ (the crack is formed by broken fibers from $-n$ to $n$). As a result, we arrive at formula (1.6), which can now be considered proved. Substitution of (1.5) and (1.6) into (1.3) leads to the numerical identity

$$\sum_{s=-n}^{n} \frac{\Gamma(n-s+3/2)\Gamma(n+s+3/2)}{\Gamma(n-s+1)\Gamma(n+s+1)} \frac{1/\pi}{(j-s)^2-1/4} = -1 + j\,\frac{\Gamma(j-n-1/2)}{\Gamma(j-n)} \frac{\Gamma(j+n+1)}{\Gamma(j+n+3/2)}.$$

**3.** We consider the case $\sigma = 0$. Instead of (1.2) and (2.3), we obtain the equalities

$$\sum_{s=L+1}^{L+M} \frac{b_s}{(j-s)^2-1/4} = P_j, \quad L+1 \leqslant j \leqslant L+M, \qquad Q_j = \sum_{s=L+1}^{L+M} \frac{b_s}{(j-s)^2-1/4}. \qquad (3.1)$$

It is assumed here that the stresses are normalized by $\max_j |P_j|$, where $L+1 \leqslant j \leqslant L+M$.

As in Sec. 2, the right side of (3.1) is a fractional rational function of $j$, whose denominator is the same polynomial of degree $M+1$ as in (2.6) and whose nominator is a polynomial of degree $M-1$ since the right side of (3.1) decreases asymptotically in proportion to $\sum_{s=L+1}^{L+M} b_s \Big/ j^2$ as $j \to \infty$. The last statement is valid if $\sum_{s=L+1}^{L+M} b_s \neq 0$ (otherwise, the decrease is faster). The order of the decrease is determined by the loads $P_j$ applied to the crack. For example, if the loads are asymmetric about the crack center, the displacements are also asymmetric and the above-mentioned sum vanishes. If the load is uniform, this sum is proportional to the work of external forces, which is equal to the elastic energy of the medium described by Eqs. (1.1) and, hence, it is nonzero. It is clear that if the load applied to the crack does not change sign, the asymptotic behavior of $Q_j$ as $j \to \infty$ is the same as in the case of one break loaded by the total force. In (1.6), we set $n = 0$ (which corresponds to one break) and subtract unity from the resulting expression (to pass from the case of remote load to the load at the break of the zeroth fiber).

Then, $Q_j = 1/(4j^2 - 1)$, i.e., the inverse square law holds asymptotically. Consequently, if the stress $P_j$ does not change sign, then $\sum_{s=L+1}^{L+M} b_s \neq 0$. Below, this condition is assumed to be satisfied.

Thus, we have

$$Q_j = \sum_{s=0}^{M-1} a_s j^s \Big/ \prod_{s=L}^{L+M} (j - s - 1/2), \tag{3.2}$$

where $a_s$ are unknown coefficients, which can be determined from the system of $M$ equalities:

$$\sum_{s=0}^{M-1} a_s j^s \Big|_{j=L+l} = P_{L+l} \prod_{s=L}^{L+M} (L + l - s - 1/2), \qquad l = 1, \dots, M. \tag{3.3}$$

From (3.3) it follows that the polynomial in the nominator of (3.2) is an interpolation polynomial with nodes $L+1, \dots, L+M$ at which it takes the values (3.3). Consequently, the nominator can be written explicitly, for example, in the Lagrange form [3, p. 35].

Let a unit stress be applied to the $r$th fiber ($L + 1 \leqslant r \leqslant L + M$) and let the other fibers be stress-free: $P_j = \delta_{jr}$ ($\delta_{jr}$ is the Kronecker delta). In this case, all loads have the same sign. As shown above, the sum is given by $\sum_{s=L+1}^{L+M} b_s \neq 0$ and representation (3.2) is valid. The interpolation polynomial now includes one term, and the stress distribution becomes

$$Q_j = \prod_{l=L}^{L+M} \frac{r - l - 1/2}{j - l - 1/2} \prod_{k=L+1,\, k\neq r}^{L+M} \frac{j - k}{r - k}. \tag{3.4}$$

For an arbitrary load, the solution is a linear combination of such solutions with various values of $r$.

In (3.4), we set $L = -n - 1$, $M = 2n + 1$, $r = 0$, and $n \geqslant 0$ (a symmetric crack is located between the $-n$th and $n$th fibers and the central fiber is loaded). In this case, the stress distribution is given by

$$Q_j = -\prod_{l=0}^{n} \frac{(l + 1/2)^2}{j^2 - (l + 1/2)^2} \prod_{l=1}^{n} \frac{j^2 - l^2}{l^2} = -\frac{1}{\pi}\Big(\frac{\Gamma(n + 3/2)}{\Gamma(n + 1)}\Big)^2 \frac{\Gamma(j + n + 1)\,\Gamma(j - n - 1/2)}{j\Gamma(j + n + 3/2)\,\Gamma(j - n)}.$$

In particular, for the first intact fiber ($j = n + 1$), we have

$$Q_{n+1} = -\frac{1}{\sqrt{\pi}}\Big(\frac{\Gamma(n + 3/2)}{\Gamma(n + 1)}\Big)^2 \frac{\Gamma(2n + 2)}{(n + 1)\,\Gamma(2n + 5/2)}.$$

One can easily verify that $Q_{n+2}/Q_{n+1} < 1$ and, hence, $Q_{n+1}$ is a decreasing function of $n$, i.e., the stress in the first fiber decreases as the crack grows under a fixed load. To break the next fiber, it is necessary to increase the load. This implies that the crack is stable in this case, unlike in the case of uniform loading [in (1.4), $Q_{n+1}$ increases with $n$].

The crack-opening displacements are obtained using system (2.5) with different right sides. In this case, to find $A_s$, we need to replace the expression under the limit sign in formula (2.4) by the right side of (3.4). As a result, we have

$$A_s = \prod_{l=L}^{L+M} (r - l - 1/2) \prod_{k=L+1,\, k\neq r}^{L+M} \frac{s - k + 1/2}{r - k} \Big/ \prod_{l=L,\, l\neq s}^{L+M} (s - l).$$

In this case, unlike in Sec. 2, the coefficients $A_s$ do not contain undetermined elements.

**4**. We assume that the breaks are grouped into several cracks and confine ourselves to the case where $P_j = 0$ and $\sigma \neq 0$. The reasoning is the same as in Sec. 2. It should be noted, however, that the degree of the polynomial in the denominator of (2.1) increases by unity each time a new crack appears. Accordingly, the degree of the polynomial in the nominator should also increase. The expression for the stress concentration is similar to (2.1), but the factor $j - a$ is replaced by the product of several factors of the form $(j - a_1), (j - a_2), (j - a_3), \dots$, which contain unknown parameters $a_1, a_2, a_3, \dots$ . The introduction of these factors leads to the required increase in the degree of the polynomial in the nominator. Instead of (2.5), we have several similar systems. Summing the equations in each of these systems, we arrive at a system of linear equations for the coefficients $a_1, a_2, a_3, \ldots$ . As a result, we have as many equations to find the solution as cracks (the number of broken fibers is of no significance). In particular, if all discontinuities are isolated, we obtain a system of the same order $M$ as the starting system (1.2).

We consider two cracks which are symmetric about the zeroth fiber. The right crack encompasses fibers $j = L+1, \ldots, L+M$. Then, Eq. (2.1) is replaced by

$$Q_j = (j^2 - a^2) \prod_{s=L+1}^{L+M} (j^2 - s^2) \Big/ \prod_{s=L}^{L+M} (j^2 - (s+1/2)^2) \tag{4.1}$$

and system (2.5) by a similar system. Owing to symmetry, the solution is expressed in terms of the single parameter $a^2$ rather than two unknowns $a_1$ and $a_2$. As a result, we obtain the explicit expression

$$a^2 = \sum_{s=L}^{L+M} (s+1/2)^2 R_s \Big/ \sum_{s=L}^{L+M} R_s,$$

$$R_s = \prod_{k=L+1}^{L+M} ((s+1/2)^2 - k^2) \Big/ (2s+1) \prod_{k=L,\, k\neq s}^{L+M} (s-k)(s+k+1). \tag{4.2}$$

From (4.2) it follows that the quantity $a^2$ is obtained by averaging the squared semi-integer variables $s+1/2$ $(L \leqslant s \leqslant L+M)$ with weights $R_s$. According to the common properties of averages, the quantity $a$ is in the range $L+1/2 \leqslant a \leqslant L+M+1/2$.

We consider the particular case of $M = 1$ [fibers with numbers $\pm(L+1)$ are broken]. Formula (4.2) implies that

$$a^2 = \frac{1}{4}\,\frac{(2L+1)(2L+3/2) + (2L+3)(2L+5/2)}{2 + 1/((2L+1)(2L+3))}.$$

Finally, if $L = 0$ and $M = 1$ (broken fibers on either side of the zeroth fiber), then $a^2 = 27/28$. In this case, we write (4.1) as

$$Q_j = (j^2 - 27/28)(j^2 - 1)/[(j^2 - 1/4)(j^2 - 9/4)] \tag{4.3}$$

and compare (4.3) with the product of the stress distributions produced by each break ignoring stress interaction [one should multiply $Q_j$ from (2.6), first setting $L = -2$ and $M = 1$ and then $L = 0$ and $M = 1$]:

$$Q_j = (j^2 - 1)(j^2 - 1)/[(j^2 - 1/4)(j^2 - 9/4)]. \tag{4.4}$$

At the site of the maximum concentration ($j = 0$), from (4.3) and (4.4) we obtain the values 12/7 and 16/9, respectively, which differ by 3.7%. As $L$ increases, the interaction between the breaks weakens and the error of this approximate calculation decreases. Generally, the stress concentration produced by several isolated breaks may also be approximated with reasonable accuracy by the product of the independent stress concentrations produced by each break.

We give another example. Let the fibers with numbers $j = -2, -1$, 1, and 2 be broken. Then, we have $L = 0$, $M = 2$, and $a^2 = 2.12$ in (4.1). In this case, the mutual interaction between the cracks is significant: the error of the formula that ignores the interaction is 6.14%.

From (2.6) and (4.1) it follows that the ratio of the exact solution to the approximate solution is equal to $(j^2 - a^2)/(j^2 - (L + (M+1)/2)^2)$ and tends to unity as the root-mean-square value of $a$ approaches the coordinate of the central point of the crack located to the right of the zeroth fiber. The latter is also valid for reasonably large absolute values of fiber numbers.

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